

A Note on the Infinitesimal Baker-Campbell-Hausdorff Formula

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Abstract

We have studied the infinitesimal Baker-Campbell-Hausdorff formula up to $n = 4$ (Math. Appl. **2** (2013), 61-91). In this note we correct some errors in our calculation for $n = 4$ and presents the calculation for $n = 5$ by using Mathematica.

1 Introduction

We are going to work within synthetic differential geometry, in which a *Lie group* G is a group and a microlinear space at the same time. For synthetic differential geometry, the reader is referred to [1] and [3]. Its *Lie algebra* (i.e., its tangent space $\mathbf{T}_e G$ of G at the identity $e \in G$), usually denoted by \mathfrak{g} , is endowed with a Lie bracket $[\cdot, \cdot]$ abiding by antisymmetry and the Jacobi identity. Each element $X \in \mathfrak{g}$ is a mapping $d \in D \mapsto X_d \in G$ with $X_0 = e$, where

$$D = \{d \in \mathbb{R} \mid d^2 = 0\}$$

We assume that the so-called *exponential mapping* $\exp : \mathfrak{g} \rightarrow G$ exists. The infinitesimal Baker-Campbell-Hausdorff formula expresses

$$\exp (d_1 + \dots + d_n) X. \exp (d_1 + \dots + d_n) Y$$

as

$$\exp (\text{a Lie polynomial of } X \text{ and } Y)$$

where $X, Y \in \mathfrak{g}$ and $d_1, \dots, d_n \in D$. In [4] we have calculated the infinitesimal Baker-Campbell-Hausdorff formula up to $n = 4$, but the second author [5] found out some errors in the calculation for $n = 4$.

This paper is based upon [5]. We correct some errors in the calculation of the infinitesimal Baker-Campbell-Hausdorff formula in case of $n = 4$ in our previous paper [4] and we present a calculation of the infinitesimal Baker-Campbell-Hausdorff formula in case of $n = 5$ newly. Both calculations were implemented by using Mathematica.

2 Preliminaries

The infinitesimal Baker-Campbell-Hausdorff formula for $n = 3$ goes as follows:

Theorem 1 (cf. Theorem 7.5 and Theorem 8.3 of [4]) *Given $X, Y \in \mathfrak{g}$ and $d_1, d_2, d_3 \in D$, we have*

$$\begin{aligned} & \exp(d_1 + d_2 + d_3) X \cdot \exp(d_1 + d_2 + d_3) Y \\ &= \exp(d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \\ & \quad \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]] \end{aligned}$$

The tangent space $\mathbf{T}_X \mathfrak{g}$ of \mathfrak{g} at $X \in \mathfrak{g}$ is naturally identified with \mathfrak{g} itself. That is to say, each $Y \in \mathfrak{g}$ gives rise to $(d \in D \mapsto X + dY \in \mathfrak{g}) \in \mathbf{T}_X \mathfrak{g}$, which yields a bijection between \mathfrak{g} and $\mathbf{T}_X \mathfrak{g}$. Its *left logarithmic derivative* $\delta^{\text{left}}(\exp)$ and its *right logarithmic derivative* $\delta^{\text{right}}(\exp)$ are characterized by the following formulas:

$$\exp X + dY = \exp X \cdot (\delta^{\text{left}}(\exp)(X)(Y))_d \quad (1)$$

and

$$\exp X + dY = (\delta^{\text{right}}(\exp)(X)(Y))_d \cdot \exp X \quad (2)$$

for any $X, Y \in \mathfrak{g}$ and any $d \in D$. For logarithmic derivatives, the reader is referred to §5 of [4] and §38.1 of [2]. We have the following well-known formulas.

Theorem 2 (cf. Theorem 5.3 and Theorem 5.8 of [4]) *Given $X \in \mathfrak{g}$ with $(\text{ad } X)^{n+1}$ vanishing for some natural number n , we have*

$$\delta^{\text{left}}(\exp)(X) = \sum_{p=0}^n \frac{(-1)^p}{(p+1)!} (\text{ad } X)^p$$

and

$$\delta^{\text{right}}(\exp)(X) = \sum_{p=0}^n \frac{1}{(p+1)!} (\text{ad } X)^p$$

We note in passing that

Proposition 3 (cf. Proposition 5.4 of [4]) *For any $X, Y \in \mathfrak{g}$ with $[X, Y]$ vanishing, we have*

$$\exp X \cdot \exp Y = \exp X + Y$$

The following simple proposition is very useful.

Proposition 4 (cf. Proposition 4.9 of [4]) For any $X \in \mathfrak{g}$ and any $d \in D$, we have

$$\exp dX = X_d$$

3 The BCH Formula for n=4

Theorem 5

$$\begin{aligned} & \exp (d_1 + d_2 + d_3 + d_4) X . \exp (d_1 + d_2 + d_3 + d_4) Y \\ &= \exp (d_1 + d_2 + d_3 + d_4) (X + Y) \\ &+ \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ &+ \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ &+ \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \end{aligned}$$

Proof. We have

$$\begin{aligned} & \exp (d_1 + d_2 + d_3 + d_4) X . \exp (d_1 + d_2 + d_3 + d_4) Y \\ &= \exp (d_1 + d_2 + d_3) X + d_4 X . \exp (d_1 + d_2 + d_3) Y + d_4 Y \\ &= \exp d_4 X . \exp (d_1 + d_2 + d_3) X . \exp (d_1 + d_2 + d_3) Y . \exp d_4 Y \\ & \text{)By Proposition 3(} \\ &= \exp d_4 X . \\ & \exp \left((d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{2} d_1 d_2 d_3 [[X, Y], Y - X] \right) . \\ & \exp d_4 Y \\ & \text{)By Theorem 1(} \end{aligned} \tag{3}$$

By the way, due to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{left}}(\exp) \left(\begin{array}{c} (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] \end{array} \right) (Y) \\
&= -\frac{1}{2}d_1[X, Y] - \frac{1}{2}d_2[X, Y] + \frac{1}{3}d_1d_2[X, [X, Y]] + \frac{1}{3}d_1d_2[Y, [X, Y]] \\
&\quad - \frac{1}{2}d_1d_2[[X, Y], Y] - \frac{1}{2}d_3[X, Y] + \frac{1}{3}d_1d_3[X, [X, Y]] + \frac{1}{3}d_1d_3[Y, [X, Y]] \\
&\quad - \frac{1}{2}d_1d_3[[X, Y], Y] + \frac{1}{3}d_2d_3[X, [X, Y]] + \frac{1}{3}d_2d_3[Y, [X, Y]] \\
&\quad - \frac{1}{2}d_2d_3[[X, Y], Y] - \frac{1}{4}d_1d_2d_3[X, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_3[X, [Y, [X, Y]]] \\
&\quad + \frac{1}{2}d_1d_2d_3[X, [[X, Y], Y]] - \frac{1}{4}d_1d_2d_3[Y, [X, [X, Y]]] \\
&\quad - \frac{1}{4}d_1d_2d_3[Y, [Y, [X, Y]]] + \frac{1}{2}d_1d_2d_3[Y, [[X, Y], Y]] \\
&\quad + \frac{1}{4}d_1d_2d_3[[[X, Y], X], Y] - \frac{1}{4}d_1d_2d_3[[[X, Y], Y], Y] + Y \tag{4}
\end{aligned}$$

Letting n_{41} be the right-hand side of (4) with the last term Y deleted, we have

$$\begin{aligned}
& (3) \\
&= \exp d_4X. \\
&\exp \left((d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] \right). \\
&\exp d_4(n_{41} + Y) \cdot \exp -d_4n_{41} \\
& \text{)By Proposition 3(} \\
&= \exp d_4X. \\
&\exp \left((d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] \right). \\
&(n_{41} + Y)_{d_4} \cdot \exp -d_4n_{41} \\
& \text{)By Proposition 4(} \\
&= \exp d_4X \cdot \exp \left(\begin{array}{c} (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4Y \end{array} \right) \cdot \exp -d_4n_{41} \\
& \text{)By (4)(} \tag{5}
\end{aligned}$$

We let i_{41} be the result of n_{41} by deleting all the terms whose coefficients contain $d_1 d_2 d_3$. Then, due to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{left}}(\exp) \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y \end{aligned} \right) (i_{41}) \\
&= \frac{1}{2}d_1 [X, Y] + \frac{1}{2}d_2 [X, Y] - \frac{5}{6}d_1 d_2 [X, [X, Y]] - \frac{5}{6}d_1 d_2 [Y, [X, Y]] \\
&+ \frac{1}{2}d_1 d_2 [[X, Y], Y] + \frac{1}{2}d_3 [X, Y] - \frac{5}{6}d_1 d_3 [X, [X, Y]] - \frac{5}{6}d_1 d_3 [Y, [X, Y]] \\
&+ \frac{1}{2}d_1 d_3 [[X, Y], Y] - \frac{5}{6}d_2 d_3 [X, [X, Y]] - \frac{5}{6}d_2 d_3 [Y, [X, Y]] \\
&+ \frac{1}{2}d_2 d_3 [[X, Y], Y] + d_1 d_2 d_3 [X, [X, [X, Y]]] \\
&+ d_1 d_2 d_3 [X, [Y, [X, Y]]] - \frac{3}{4}d_1 d_2 d_3 [X, [[X, Y], Y]] + d_1 d_2 d_3 [Y, [X, [X, Y]]] \\
&+ d_1 d_2 d_3 [Y, [Y, [X, Y]]] - \frac{3}{4}d_1 d_2 d_3 [Y, [[X, Y], Y]] \tag{6}
\end{aligned}$$

Letting n_{42} be the right-hand side of (6), we have

$$\begin{aligned}
& (5) \\
&= \exp d_4 X. \exp \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y \end{aligned} \right). \\
&\exp d_4 n_{42}. \exp -d_4 n_{42} - d_4 n_{41} \\
& \text{)By Proposition 3(} \\
&= \exp d_4 X. \exp \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y \end{aligned} \right). \\
&(n_{42})_{d_4}. \exp -d_4 n_{42} - d_4 n_{41} \\
& \text{)By Proposition 4(} \\
&= \exp d_4 X. \exp \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} \end{aligned} \right). \\
&\exp -d_4 n_{42} - d_4 n_{41} \\
& \text{)By (6)(} \tag{7}
\end{aligned}$$

We let i_{42} be the result of $-n_{42} - n_{41}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3$. Then, thanks to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{left}}(\text{exp}) \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} \end{aligned} \right) (i_{42}) \\
&= \frac{1}{2}d_1 d_2 [X, [X, Y]] + \frac{1}{2}d_1 d_2 [Y, [X, Y]] + \frac{1}{2}d_1 d_3 [X, [X, Y]] \\
&+ \frac{1}{2}d_1 d_3 [Y, [X, Y]] + \frac{1}{2}d_2 d_3 [X, [X, Y]] + \frac{1}{2}d_2 d_3 [Y, [X, Y]] \\
&- \frac{3}{4}d_1 d_2 d_3 [X, [X, [X, Y]]] - \frac{3}{4}d_1 d_2 d_3 [X, [Y, [X, Y]]] \\
&- \frac{3}{4}d_1 d_2 d_3 [Y, [X, [X, Y]]] - \frac{3}{4}d_1 d_2 d_3 [Y, [Y, [X, Y]]] \tag{8}
\end{aligned}$$

Letting n_{43} be the right-hand side of (8), we have

$$\begin{aligned}
& (7) \\
&= \exp d_4 X. \exp \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} \end{aligned} \right). \\
&\exp d_4 n_{43}. \exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
& \text{)By Proposition 3(} \\
&= \exp d_4 X. \exp \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} \end{aligned} \right). \\
&(n_{43})_{d_4}. \exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
& \text{)By Proposition 4(} \\
&= \exp d_4 X. \exp \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} + d_4 i_{42} \end{aligned} \right). \\
&\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
& \text{)By (8)(} \tag{9}
\end{aligned}$$

Since the coefficient of every term in $-n_{43} - n_{42} - n_{41}$ contains $d_1 d_2 d_3$, we now turn our attention to the left $\exp d_4 X$. Now, thanks to Theorem 2, we

have

$$\begin{aligned}
& \delta^{\text{right}}(\exp) \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4Y + d_4i_{41} + d_4i_{42} \end{aligned} \right) (X) \\
&= -\frac{1}{2}d_1[X, Y] - \frac{1}{2}d_2[X, Y] - \frac{1}{3}d_1d_2[X, [X, Y]] - \frac{1}{3}d_1d_2[Y, [X, Y]] \\
&+ \frac{1}{2}d_1d_2[[X, Y], X] - \frac{1}{2}d_3[X, Y] - \frac{1}{3}d_1d_3[X, [X, Y]] - \frac{1}{3}d_1d_3[Y, [X, Y]] \\
&+ \frac{1}{2}d_1d_3[[X, Y], X] - \frac{1}{3}d_2d_3[X, [X, Y]] - \frac{1}{3}d_2d_3[Y, [X, Y]] \\
&+ \frac{1}{2}d_2d_3[[X, Y], X] - \frac{1}{4}d_1d_2d_3[X, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_3[X, [Y, [X, Y]]] \\
&+ \frac{1}{2}d_1d_2d_3[X, [[X, Y], X]] - \frac{1}{4}d_1d_2d_3[Y, [X, [X, Y]]] \\
&- \frac{1}{4}d_1d_2d_3[Y, [Y, [X, Y]]] + \frac{1}{2}d_1d_2d_3[Y, [[X, Y], X]] \\
&- \frac{1}{2}d_1d_2d_3[[X, Y], [X, Y]] - \frac{1}{4}d_1d_2d_3[[[X, Y], X], X] \\
&+ \frac{1}{4}d_1d_2d_3[[[X, Y], Y], X] + X
\end{aligned} \tag{10}$$

We let m_{41} be the right-hand side of (10) with the last term X deleted. Then we have

$$\begin{aligned}
& (9) \\
&= \exp -d_4m_{41} \cdot \exp d_4X + d_4m_{41} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4Y + d_4i_{41} + d_4i_{42} \end{aligned} \right) \cdot \\
& \exp -d_4n_{43} - d_4n_{42} - d_4n_{41} \\
&) \text{By Proposition 3(} \\
&= \exp -d_4m_{41} \cdot (X + m_{41})_{d_4} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4Y + d_4i_{41} + d_4i_{42} \end{aligned} \right) \cdot \\
& \exp -d_4n_{43} - d_4n_{42} - d_4n_{41} \\
&) \text{By Proposition 4(} \\
&= \exp -d_4m_{41} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4i_{41} + d_4i_{42} \end{aligned} \right) \cdot \\
& \exp -d_4n_{43} - d_4n_{42} - d_4n_{41} \\
&) \text{By (10)(}
\end{aligned} \tag{11}$$

We let j_{41} be the result of $-m_{41}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3$. Then, thanks to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{right}}(\exp) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 i_{42} \end{aligned} \right) (j_{41}) \\
&= \frac{1}{2}d_1 [X, Y] + \frac{1}{2}d_2 [X, Y] + \frac{5}{6}d_1 d_2 [X, [X, Y]] + \frac{5}{6}d_1 d_2 [Y, [X, Y]] \\
&- \frac{1}{2}d_1 d_2 [[X, Y], X] + \frac{1}{2}d_3 [X, Y] + \frac{5}{6}d_1 d_3 [X, [X, Y]] \\
&+ \frac{5}{6}d_1 d_3 [Y, [X, Y]] - \frac{1}{2}d_1 d_3 [[X, Y], X] + \frac{5}{6}d_2 d_3 [X, [X, Y]] \\
&+ \frac{5}{6}d_2 d_3 [Y, [X, Y]] - \frac{1}{2}d_2 d_3 [[X, Y], X] + d_1 d_2 d_3 [X, [X, [X, Y]]] \\
&+ d_1 d_2 d_3 [X, [Y, [X, Y]]] - \frac{3}{4}d_1 d_2 d_3 [X, [[X, Y], X]] \\
&+ d_1 d_2 d_3 [Y, [X, [X, Y]]] + d_1 d_2 d_3 [Y, [Y, [X, Y]]] \\
&- \frac{3}{4}d_1 d_2 d_3 [Y, [[X, Y], X]] + \frac{3}{4}d_1 d_2 d_3 [[X, Y], [X, Y]] \tag{12}
\end{aligned}$$

Letting m_{42} be the right-hand side of (12), we have

$$\begin{aligned}
& (11) \\
&= \exp -d_4 m_{41} - d_4 m_{42} \cdot \exp d_4 m_{42} \cdot \\
&\exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 i_{42} \end{aligned} \right) \cdot \\
&\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
&)\text{By Proposition 3(} \\
&= \exp -d_4 m_{41} - d_4 m_{42} \cdot (m_{42})_{d_4} \cdot \\
&\exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 i_{42} \end{aligned} \right) \cdot \\
&\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
&)\text{By Proposition 4(} \\
&= \exp -d_4 m_{41} - d_4 m_{42} \cdot \\
&\exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42} \end{aligned} \right) \cdot \\
&\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
&)\text{By (12)(} \tag{13}
\end{aligned}$$

We let j_{42} be the result of $-m_{41} - m_{42}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3$. Then, thanks to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{right}}(\exp) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42} \end{aligned} \right) (j_{42}) \\
&= -\frac{1}{2}d_1 d_2 [X, [X, Y]] - \frac{1}{2}d_1 d_2 [Y, [X, Y]] - \frac{1}{2}d_1 d_3 [X, [X, Y]] \\
&- \frac{1}{2}d_1 d_3 [Y, [X, Y]] - \frac{1}{2}d_2 d_3 [X, [X, Y]] - \frac{1}{2}d_2 d_3 [Y, [X, Y]] \\
&- \frac{3}{4}d_1 d_2 d_3 [X, [X, [X, Y]]] - \frac{3}{4}d_1 d_2 d_3 [X, [Y, [X, Y]]] \\
&- \frac{3}{4}d_1 d_2 d_3 [Y, [X, [X, Y]]] - \frac{3}{4}d_1 d_2 d_3 [Y, [Y, [X, Y]]] \tag{14}
\end{aligned}$$

Letting m_{43} be the right-hand side of (14), we have

$$\begin{aligned}
& (13) \\
&= \exp -d_4 m_{41} - d_4 m_{42} - d_4 m_{43} \cdot \exp d_4 m_{43} \cdot \\
&\exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42} \end{aligned} \right) \cdot \\
&\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
&)\text{By Proposition 3(} \\
&= \exp -d_4 m_{41} - d_4 m_{42} - d_4 m_{43} \cdot (m_{43})_{d_4} \\
&\exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42} \end{aligned} \right) \cdot \\
&\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
&)\text{By Proposition 4(} \\
&= \exp -d_4 m_{41} - d_4 m_{42} - d_4 m_{43} \cdot \\
&\exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42} + d_4 j_{42} \end{aligned} \right) \cdot \\
&\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \\
&)\text{By (14)(} \tag{15}
\end{aligned}$$

Since the coefficient of every term in $-m_{41} - m_{42} - m_{43}$ contains $d_1 d_2 d_3$, we are done, so that we have

$$\begin{aligned}
& (15) \\
&= \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] \\ & + \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42} + d_4 j_{42} \\ & - d_4 m_{41} - d_4 m_{42} - d_4 m_{43} - d_4 n_{43} - d_4 n_{42} - d_4 n_{41} \end{aligned} \right) \tag{16}
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
& j_{41} + i_{42} + j_{42} \\
&= d_1 [X, Y] + d_2 [X, Y] - \frac{1}{2} d_1 d_2 [[X, Y], X] + \frac{1}{2} d_1 d_2 [[X, Y], Y] \\
& d_3 [X, Y] - \frac{1}{2} d_1 d_3 [[X, Y], X] + \frac{1}{2} d_1 d_3 [[X, Y], Y] \\
& - \frac{1}{2} d_2 d_3 [[X, Y], X] + \frac{1}{2} d_2 d_3 [[X, Y], Y]
\end{aligned}$$

whereas

$$\begin{aligned}
& -m_{41} - m_{42} - m_{43} - n_{43} - n_{42} - n_{41} \\
&= \frac{1}{4} d_1 d_2 d_3 [X, [[X, Y], X]] + \frac{1}{4} d_1 d_2 d_3 [X, [[X, Y], Y]] \\
& + \frac{1}{4} d_1 d_2 d_3 [Y, [[X, Y], X]] + \frac{1}{4} d_1 d_2 d_3 [Y, [[X, Y], Y]] \\
& + \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], X] - \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], Y] \\
& - \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], X] + \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], Y]
\end{aligned}$$

Therefore we have the desired result. ■

4 The BCH Formula for n=5

Theorem 6

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3 + d_4 + d_5) X \cdot \exp (d_1 + d_2 + d_3 + d_4 + d_5) Y \\
&= \exp (d_1 + d_2 + d_3 + d_4 + d_5) (X + Y) \\
& + \frac{1}{2} (d_1 + d_2 + d_3 + d_4 + d_5)^2 [X, Y] \\
& + \frac{1}{12} (d_1 + d_2 + d_3 + d_4 + d_5)^3 [[X, Y], Y - X] \\
& + \frac{1}{96} (d_1 + d_2 + d_3 + d_4 + d_5)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\
& + \frac{1}{120} (d_1 + d_2 + d_3 + d_4 + d_5)^5 \left(\begin{aligned} & \frac{5}{6} [X + Y, [X + Y, [[X, Y], X - Y]] \\ & + \frac{1}{2} [[X, Y], [[X, Y], X + Y]] \\ & + \frac{1}{8} [[X + Y, [[X, Y], X + Y]], Y - X] \\ & + \frac{1}{8} [[[X, Y], Y - X], X - Y], X - Y \end{aligned} \right)
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& \exp (d_1 + d_2 + d_3 + d_4 + d_5) X . \exp (d_1 + d_2 + d_3 + d_4 + d_5) Y \\
&= \exp (d_1 + d_2 + d_3 + d_4) X + d_5 X . \exp (d_1 + d_2 + d_3 + d_4) Y + d_5 Y \\
&= \exp d_5 X . \exp (d_1 + d_2 + d_3 + d_4) X . \exp (d_1 + d_2 + d_3 + d_4) Y . \exp d_5 Y \\
& \text{)By Proposition 3(} \\
&= \exp d_5 X . \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \end{aligned} \right) . \\
& \exp d_5 Y \\
& \text{)By Theorem 5(} \tag{17}
\end{aligned}$$

By the way, due to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{left}}(\text{exp}) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4[X + Y, [[X, Y], X + Y]] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4[[[X, Y], X - Y], X - Y] \end{aligned} \right) (Y) \\
&= Y - \frac{1}{2}d_1[X, Y] - \frac{1}{2}d_2[X, Y] + \frac{1}{3}d_1d_2[X, [X, Y]] + \frac{1}{3}d_1d_2[Y, [X, Y]] \\
&\quad - \frac{1}{2}d_1d_2[[X, Y], Y] - \frac{1}{2}d_3[X, Y] + \frac{1}{3}d_1d_3[X, [X, Y]] + \frac{1}{3}d_1d_3[Y, [X, Y]] \\
&\quad - \frac{1}{2}d_1d_3[[X, Y], Y] + \frac{1}{3}d_2d_3[X, [X, Y]] + \frac{1}{3}d_2d_3[Y, [X, Y]] - \frac{1}{2}d_2d_3[[X, Y], Y] \\
&\quad - \frac{1}{4}d_1d_2d_3[X, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_3[X, [Y, [X, Y]]] + \frac{1}{2}d_1d_2d_3[X, [[X, Y], Y]] \\
&\quad - \frac{1}{4}d_1d_2d_3[Y, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_3[Y, [Y, [X, Y]]] + \frac{1}{2}d_1d_2d_3[Y, [[X, Y], Y]] \\
&\quad + \frac{1}{4}d_1d_2d_3[[[X, Y], X], Y] - \frac{1}{4}d_1d_2d_3[[[X, Y], Y], Y] - \frac{1}{2}d_4[X, Y] + \frac{1}{3}d_1d_4[X, [X, Y]] \\
&\quad + \frac{1}{3}d_1d_4[Y, [X, Y]] - \frac{1}{2}d_1d_4[[X, Y], Y] + \frac{1}{3}d_2d_4[X, [X, Y]] + \frac{1}{3}d_2d_4[Y, [X, Y]] \\
&\quad - \frac{1}{2}d_2d_4[[X, Y], Y] - \frac{1}{4}d_1d_2d_4[X, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_4[X, [Y, [X, Y]]] \\
&\quad + \frac{1}{2}d_1d_2d_4[X, [[X, Y], Y]] - \frac{1}{4}d_1d_2d_4[Y, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_4[Y, [Y, [X, Y]]] \\
&\quad + \frac{1}{2}d_1d_2d_4[Y, [[X, Y], Y]] + \frac{1}{4}d_1d_2d_4[[[X, Y], X], Y] - \frac{1}{4}d_1d_2d_4[[[X, Y], Y], Y] \\
&\quad + \frac{1}{3}d_3d_4[X, [X, Y]] + \frac{1}{3}d_3d_4[Y, [X, Y]] - \frac{1}{2}d_3d_4[[X, Y], Y] - \frac{1}{4}d_1d_3d_4[X, [X, [X, Y]]] \\
&\quad - \frac{1}{4}d_1d_3d_4[X, [Y, [X, Y]]] + \frac{1}{2}d_1d_3d_4[X, [[X, Y], Y]] - \frac{1}{4}d_1d_3d_4[Y, [X, [X, Y]]] \\
&\quad - \frac{1}{4}d_1d_3d_4[Y, [Y, [X, Y]]] + \frac{1}{2}d_1d_3d_4[Y, [[X, Y], Y]] + \frac{1}{4}d_1d_3d_4[[[X, Y], X], Y] \\
&\quad - \frac{1}{4}d_1d_3d_4[[[X, Y], Y], Y] - \frac{1}{4}d_2d_3d_4[X, [X, [X, Y]]] - \frac{1}{4}d_2d_3d_4[X, [Y, [X, Y]]] \\
&\quad + \frac{1}{2}d_2d_3d_4[X, [[X, Y], Y]] - \frac{1}{4}d_2d_3d_4[Y, [X, [X, Y]]] - \frac{1}{4}d_2d_3d_4[Y, [Y, [X, Y]]] \\
&\quad + \frac{1}{2}d_2d_3d_4[Y, [[X, Y], Y]] + \frac{1}{4}d_2d_3d_4[[[X, Y], X], Y] - \frac{1}{4}d_2d_3d_4[[[X, Y], Y], Y] \\
&\quad + \frac{1}{5}d_1d_2d_3d_4[X, [X, [X, [X, Y]]]] + \frac{1}{5}d_1d_2d_3d_4[X, [X, [Y, [X, Y]]]] \\
&\quad - \frac{1}{2}d_1d_2d_3d_4[X, [X, [[X, Y], Y]]] + \frac{1}{5}d_1d_2d_3d_4[X, [Y, [X, [X, Y]]]] \\
&\quad + \frac{1}{5}d_1d_2d_3d_4[X, [Y, [Y, [X, Y]]]] - \frac{1}{2}d_1d_2d_3d_4[X, [Y, [[X, Y], Y]]] \\
&\quad - \frac{1}{3}d_1d_2d_3d_4[X, [[[X, Y], X], Y]] + \frac{1}{3}d_1d_2d_3d_4[X, [[[X, Y], Y], Y]]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{5}d_1d_2d_3d_4[Y, [X, [X, [X, Y]]]] + \frac{1}{5}d_1d_2d_3d_4[Y, [X, [Y, [X, Y]]]] \\
& - \frac{1}{2}d_1d_2d_3d_4[Y, [X, [[X, Y], Y]]] + \frac{1}{5}d_1d_2d_3d_4[Y, [Y, [X, [X, Y]]]] \\
& + \frac{1}{5}d_1d_2d_3d_4[Y, [Y, [Y, [X, Y]]]] - \frac{1}{2}d_1d_2d_3d_4[Y, [Y, [[X, Y], Y]]] \\
& - \frac{1}{3}d_1d_2d_3d_4[Y, [[[X, Y], X], Y]] + \frac{1}{3}d_1d_2d_3d_4[Y, [[[X, Y], Y], Y]] \\
& - \frac{1}{2}d_1d_2d_3d_4[[X, Y], [X, [X, Y]]] - \frac{1}{2}d_1d_2d_3d_4[[X, Y], [Y, [X, Y]]] \\
& + d_1d_2d_3d_4[[X, Y], [[X, Y], Y]] - \frac{1}{8}d_1d_2d_3d_4[[X, [[X, Y], X]], Y] \\
& - \frac{1}{8}d_1d_2d_3d_4[[X, [[X, Y], Y]], Y] - \frac{1}{8}d_1d_2d_3d_4[[Y, [[X, Y], X]], Y] \\
& - \frac{1}{8}d_1d_2d_3d_4[[Y, [[X, Y], Y]], Y] - \frac{1}{3}d_1d_2d_3d_4[[[X, Y], X], [X, Y]] \\
& + \frac{1}{3}d_1d_2d_3d_4[[[X, Y], Y], [X, Y]] - \frac{1}{8}d_1d_2d_3d_4[[[[X, Y], X], X], Y] \\
& + \frac{1}{8}d_1d_2d_3d_4[[[[X, Y], X], Y], Y] + \frac{1}{8}d_1d_2d_3d_4[[[[X, Y], Y], X], Y] \\
& - \frac{1}{8}d_1d_2d_3d_4[[[[X, Y], Y], Y], Y] \tag{18}
\end{aligned}$$

Letting n_{51} be the right-hand side of (18) with the first term Y deleted, we have

$$\begin{aligned}
& (17) \\
& = \exp d_5X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \end{aligned} \right). \\
& \exp d_5X + d_5n_{51}. \exp -d_5n_{51} \\
&) \text{By Proposition 3(} \\
& = \exp d_5X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \end{aligned} \right). \\
& \exp -d_5n_{51} \\
&) \text{By (18)(} \tag{19}
\end{aligned}$$

We let i_{51} be the result of $-n_{51}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, by dint of Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{left}}(\text{exp}) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 [X + Y, [[X, Y], X + Y]] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 [[[X, Y], X - Y], X - Y] \end{aligned} \right) (i_{51}) \\
&= \frac{1}{2}d_1 [X, Y] + \frac{1}{2}d_2 [X, Y] - \frac{5}{6}d_1 d_2 [X, [X, Y]] - \frac{5}{6}d_1 d_2 [Y, [X, Y]] + \frac{1}{2}d_1 d_2 [[X, Y], Y] \\
&+ \frac{1}{2}d_3 [X, Y] - \frac{5}{6}d_1 d_3 [X, [X, Y]] - \frac{5}{6}d_1 d_3 [Y, [X, Y]] + \frac{1}{2}d_1 d_3 [[X, Y], Y] \\
&- \frac{5}{6}d_2 d_3 [X, [X, Y]] - \frac{5}{6}d_2 d_3 [Y, [X, Y]] + \frac{1}{2}d_2 d_3 [[X, Y], Y] + \frac{5}{4}d_1 d_2 d_3 [X, [X, [X, Y]]] \\
&+ \frac{5}{4}d_1 d_2 d_3 [X, [Y, [X, Y]]] - \frac{5}{4}d_1 d_2 d_3 [X, [[X, Y], Y]] + \frac{5}{4}d_1 d_2 d_3 [Y, [X, [X, Y]]] \\
&+ \frac{5}{4}d_1 d_2 d_3 [Y, [Y, [X, Y]]] - \frac{5}{4}d_1 d_2 d_3 [Y, [[X, Y], Y]] - \frac{1}{4}d_1 d_2 d_3 [[[X, Y], X], Y] \\
&+ \frac{1}{4}d_1 d_2 d_3 [[[X, Y], Y], Y] + \frac{1}{2}d_4 [X, Y] - \frac{5}{6}d_1 d_4 [X, [X, Y]] - \frac{5}{6}d_1 d_4 [Y, [X, Y]] \\
&+ \frac{1}{2}d_1 d_4 [[X, Y], Y] - \frac{5}{6}d_2 d_4 [X, [X, Y]] - \frac{5}{6}d_2 d_4 [Y, [X, Y]] + \frac{1}{2}d_2 d_4 [[X, Y], Y] \\
&+ \frac{5}{4}d_1 d_2 d_4 [X, [X, [X, Y]]] + \frac{5}{4}d_1 d_2 d_4 [X, [Y, [X, Y]]] - \frac{5}{4}d_1 d_2 d_4 [X, [[X, Y], Y]] \\
&+ \frac{5}{4}d_1 d_2 d_4 [Y, [X, [X, Y]]] + \frac{5}{4}d_1 d_2 d_4 [Y, [Y, [X, Y]]] - \frac{5}{4}d_1 d_2 d_4 [Y, [[X, Y], Y]] \\
&- \frac{1}{4}d_1 d_2 d_4 [[[X, Y], X], Y] + \frac{1}{4}d_1 d_2 d_4 [[[X, Y], Y], Y] - \frac{5}{6}d_3 d_4 [X, [X, Y]] \\
&- \frac{5}{6}d_3 d_4 [Y, [X, Y]] + \frac{1}{2}d_3 d_4 [[X, Y], X] + \frac{5}{4}d_1 d_3 d_4 [X, [X, [X, Y]]] \\
&+ \frac{5}{4}d_1 d_3 d_4 [X, [Y, [X, Y]]] - \frac{5}{4}d_1 d_3 d_4 [X, [[X, Y], Y]] + \frac{5}{4}d_1 d_3 d_4 [Y, [X, [X, Y]]] \\
&+ \frac{5}{4}d_1 d_3 d_4 [Y, [Y, [X, Y]]] - \frac{5}{4}d_1 d_3 d_4 [Y, [[X, Y], Y]] - \frac{1}{4}d_1 d_3 d_4 [[[X, Y], X], Y] \\
&+ \frac{1}{4}d_1 d_3 d_4 [[[X, Y], Y], Y] + \frac{5}{4}d_2 d_3 d_4 [X, [X, [X, Y]]] + \frac{5}{4}d_2 d_3 d_4 [X, [Y, [X, Y]]] \\
&- \frac{5}{4}d_2 d_3 d_4 [X, [[X, Y], Y]] + \frac{5}{4}d_2 d_3 d_4 [Y, [X, [X, Y]]] + \frac{5}{4}d_2 d_3 d_4 [Y, [Y, [X, Y]]] \\
&- \frac{5}{4}d_2 d_3 d_4 [Y, [[X, Y], Y]] - \frac{1}{4}d_2 d_3 d_4 [[[X, Y], X], Y] + \frac{1}{4}d_2 d_3 d_4 [[[X, Y], Y], Y] \\
&- \frac{5}{3}d_1 d_2 d_3 d_4 [X, [X, [X, [X, Y]]]] - \frac{5}{3}d_1 d_2 d_3 d_4 [X, [X, [Y, [X, Y]]]] \\
&+ 2d_1 d_2 d_3 d_4 [X, [X, [[X, Y], Y]]] - \frac{5}{3}d_1 d_2 d_3 d_4 [X, [Y, [X, [X, Y]]]] \\
&- \frac{5}{3}d_1 d_2 d_3 d_4 [X, [Y, [Y, [X, Y]]]] + 2d_1 d_2 d_3 d_4 [X, [Y, [[X, Y], Y]]] \\
&+ \frac{1}{2}d_1 d_2 d_3 d_4 [X, [[[X, Y], X], Y]] - \frac{1}{2}d_1 d_2 d_3 d_4 [X, [[[X, Y], Y], Y]]
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{3}d_1d_2d_3d_4[Y, [X, [X, [X, Y]]]] - \frac{5}{3}d_1d_2d_3d_4[Y, [X, [Y, [X, Y]]]] \\
& + 2d_1d_2d_3d_4[Y, [X, [[X, Y], Y]]] - \frac{5}{3}d_1d_2d_3d_4[Y, [Y, [X, [X, Y]]]] \\
& - \frac{5}{3}d_1d_2d_3d_4[Y, [Y, [Y, [X, Y]]]] + 2d_1d_2d_3d_4[Y, [Y, [[X, Y], Y]]] \\
& + \frac{1}{2}d_1d_2d_3d_4[Y, [[[X, Y], X], Y]] - \frac{1}{2}d_1d_2d_3d_4[Y, [[[X, Y], Y], Y]] \\
& + 2d_1d_2d_3d_4[[X, Y], [X, [X, Y]]] + 2d_1d_2d_3d_4[[X, Y], [Y, [X, Y]]] \\
& - \frac{3}{2}d_1d_2d_3d_4[[X, Y], [[X, Y], Y]] + \frac{1}{2}d_1d_2d_3d_4[[[X, Y], X], [X, Y]] \\
& - \frac{1}{2}d_1d_2d_3d_4[[[X, Y], Y], [X, Y]]
\end{aligned} \tag{20}$$

Letting n_{52} be the right-hand side of (20), we have

$$\begin{aligned}
& (19) \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \end{aligned} \right). \\
& \exp d_5 n_{52} \cdot \exp -d_5 n_{51} - d_5 n_{52} \\
&) \text{By Proposition 3(} \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \end{aligned} \right). \\
& (n_{52})_{d_5} \cdot \exp -d_5 n_{51} - d_5 n_{52} \\
&) \text{By Proposition 4(} \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} \end{aligned} \right). \\
& \exp -d_5 n_{51} - d_5 n_{52} \\
&) \text{By (20)(} \tag{21}
\end{aligned}$$

We let i_{52} be the result of $-n_{51} - n_{52}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, thanks to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{left}}(\text{exp}) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 [X + Y, [[X, Y], X + Y]] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 [[[X, Y], X - Y], X - Y] + d_5 i_{51} \end{aligned} \right) (i_{52}) \\
&= \frac{1}{2} d_1 d_2 [X, [X, Y]] + \frac{1}{2} d_1 d_2 [Y, [X, Y]] + \frac{1}{2} d_1 d_3 [X, [X, Y]] + \frac{1}{2} d_1 d_3 [Y, [X, Y]] \\
&+ \frac{1}{2} d_2 d_3 [X, [X, Y]] + \frac{1}{2} d_2 d_3 [Y, [X, Y]] - \frac{7}{4} d_1 d_2 d_3 [X, [X, [X, Y]]] \\
&- \frac{7}{4} d_1 d_2 d_3 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [X, [[X, Y], Y]] - \frac{7}{4} d_1 d_2 d_3 [Y, [X, [X, Y]]] \\
&- \frac{7}{4} d_1 d_2 d_3 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [Y, [[X, Y], Y]] + \frac{1}{2} d_1 d_4 [X, [X, Y]] \\
&+ \frac{1}{2} d_1 d_4 [Y, [X, Y]] + \frac{1}{2} d_2 d_4 [X, [X, Y]] + \frac{1}{2} d_2 d_4 [Y, [X, Y]] - \frac{7}{4} d_1 d_2 d_4 [X, [X, [X, Y]]] \\
&- \frac{7}{4} d_1 d_2 d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [[X, Y], Y]] - \frac{7}{4} d_1 d_2 d_4 [Y, [X, [X, Y]]] \\
&- \frac{7}{4} d_1 d_2 d_4 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [Y, [[X, Y], Y]] + \frac{1}{2} d_3 d_4 [X, [X, Y]] \\
&+ \frac{1}{2} d_3 d_4 [Y, [X, Y]] - \frac{7}{4} d_1 d_3 d_4 [X, [X, [X, Y]]] - \frac{7}{4} d_1 d_3 d_4 [X, [Y, [X, Y]]] \\
&+ \frac{3}{4} d_1 d_3 d_4 [X, [[X, Y], Y]] - \frac{7}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]] - \frac{7}{4} d_1 d_3 d_4 [Y, [Y, [X, Y]]] \\
&+ \frac{3}{4} d_1 d_3 d_4 [Y, [[X, Y], Y]] - \frac{7}{4} d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{7}{4} d_2 d_3 d_4 [X, [Y, [X, Y]]] \\
&+ \frac{3}{4} d_2 d_3 d_4 [X, [[X, Y], Y]] - \frac{7}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]] - \frac{7}{4} d_2 d_3 d_4 [Y, [Y, [X, Y]]] \\
&+ \frac{3}{4} d_2 d_3 d_4 [Y, [[X, Y], Y]] + 3 d_1 d_2 d_3 d_4 [X, [X, [X, [X, Y]]]] \\
&+ 3 d_1 d_2 d_3 d_4 [X, [X, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [[X, Y], Y]]] \\
&+ 3 d_1 d_2 d_3 d_4 [X, [Y, [X, [X, Y]]]] + 3 d_1 d_2 d_3 d_4 [X, [Y, [Y, [X, Y]]]] \\
&- \frac{1}{2} d_1 d_2 d_3 d_4 [X, [Y, [[X, Y], Y]]] + 3 d_1 d_2 d_3 d_4 [Y, [X, [X, [X, Y]]]] \\
&+ 3 d_1 d_2 d_3 d_4 [Y, [X, [Y, [X, Y]]]] - \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [[X, Y], Y]]] \\
&+ 3 d_1 d_2 d_3 d_4 [Y, [Y, [X, [X, Y]]]] + 3 d_1 d_2 d_3 d_4 [Y, [Y, [Y, [X, Y]]]] \\
&- \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [Y, [[X, Y], Y]]] - \frac{3}{2} d_1 d_2 d_3 d_4 [[[X, Y], [X, [X, Y]]] \\
&- \frac{3}{2} d_1 d_2 d_3 d_4 [[[X, Y], [Y, [X, Y]]] \tag{22}
\end{aligned}$$

Letting n_{53} be the right-hand side of (22), we have

$$\begin{aligned}
& (21) \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} \end{aligned} \right). \\
& \exp d_5 n_{53} \cdot \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} \\
&) \text{By Proposition 3(} \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} \end{aligned} \right). \\
& (n_{53})_{d_5} \cdot \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} \\
&) \text{By Proposition 4(} \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} \end{aligned} \right). \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} \\
&) \text{By (22)(} \tag{23}
\end{aligned}$$

We let i_{53} be the result of $-n_{51} - n_{52} - n_{53}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, due to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{left}}(\text{exp}) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 [X + Y, [[X, Y], X + Y]] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 [[[X, Y], X - Y], X - Y] \\ & + d_5 i_{51} + d_5 i_{52} \end{aligned} \right) (i_{53}) \\
&= \frac{3}{4} d_1 d_2 d_3 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [Y, [X, [X, Y]]] \\
&+ \frac{3}{4} d_1 d_2 d_3 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [Y, [X, Y]]] \\
&+ \frac{3}{4} d_1 d_2 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [X, [X, [X, Y]]] \\
&+ \frac{3}{4} d_1 d_3 d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [Y, [Y, [X, Y]]] \\
&+ \frac{3}{4} d_2 d_3 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]] \\
&+ \frac{3}{4} d_2 d_3 d_4 [Y, [Y, [X, Y]]] - \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [X, [X, Y]]]] \\
&- \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [Y, [X, Y]]]] - \frac{1}{2} d_1 d_2 d_3 d_4 [X, [Y, [X, [X, Y]]]] \\
&- \frac{1}{2} d_1 d_2 d_3 d_4 [X, [Y, [Y, [X, Y]]]] - \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [X, [X, Y]]]] \\
&- \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [Y, [X, Y]]]] - \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [Y, [X, [X, Y]]]] \\
&- \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [Y, [Y, [X, Y]]]] \tag{24}
\end{aligned}$$

Letting n_{54} be the right-hand side of (24), we have

$$\begin{aligned}
& (23) \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} \end{aligned} \right). \\
& \exp d_5 n_{54} \cdot \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By Proposition 3(} \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} \end{aligned} \right). \\
& (n_{54})_{d_5} \cdot \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By Proposition 4(} \\
& = \exp d_5 X. \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} \end{aligned} \right). \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By (24)(} \tag{25}
\end{aligned}$$

Since the coefficient of every term in $-n_{51}-n_{52}-n_{53}-n_{54}$ contains $d_1d_2d_3d_4$, we turn our attention to the left $\exp d_5X$. Now, thanks to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{right}}(\exp) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4[X + Y, [[X, Y], X + Y]] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4[[[X, Y], X - Y], X - Y] \\ & + d_5i_{51} + d_5i_{52} + d_5i_{53} \end{aligned} \right) (X) \\
&= X - \frac{1}{2}d_1[X, Y] - \frac{1}{2}d_2[X, Y] - \frac{1}{3}d_1d_2[X, [X, Y]] - \frac{1}{3}d_1d_2[Y, [X, Y]] \\
&+ \frac{1}{2}d_1d_2[[X, Y], X] - \frac{1}{2}d_3[X, Y] - \frac{1}{3}d_1d_3[X, [X, Y]] - \frac{1}{3}d_1d_3[Y, [X, Y]] \\
&+ \frac{1}{2}d_1d_3[[X, Y], X] - \frac{1}{3}d_2d_3[X, [X, Y]] - \frac{1}{3}d_2d_3[Y, [X, Y]] + \frac{1}{2}d_2d_3[[X, Y], X] \\
&- \frac{1}{4}d_1d_2d_3[X, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_3[X, [Y, [X, Y]]] + \frac{1}{2}d_1d_2d_3[X, [[X, Y], X]] \\
&- \frac{1}{4}d_1d_2d_3[Y, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_3[Y, [Y, [X, Y]]] + \frac{1}{2}d_1d_2d_3[Y, [[X, Y], X]] \\
&- \frac{1}{4}d_1d_2d_3[[[X, Y], X], X] + \frac{1}{4}d_1d_2d_3[[[X, Y], Y], X] - \frac{1}{2}d_4[X, Y] - \frac{1}{3}d_1d_4[X, [X, Y]] \\
&- \frac{1}{3}d_1d_4[Y, [X, Y]] + \frac{1}{2}d_1d_4[[X, Y], X] - \frac{1}{3}d_2d_4[X, [X, Y]] - \frac{1}{3}d_2d_4[Y, [X, Y]] \\
&+ \frac{1}{2}d_2d_4[[X, Y], X] - \frac{1}{4}d_1d_2d_4[X, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_4[X, [Y, [X, Y]]] \\
&+ \frac{1}{2}d_1d_2d_4[X, [[X, Y], X]] - \frac{1}{4}d_1d_2d_4[Y, [X, [X, Y]]] - \frac{1}{4}d_1d_2d_4[Y, [Y, [X, Y]]] \\
&+ \frac{1}{2}d_1d_2d_4[Y, [[X, Y], X]] - \frac{1}{4}d_1d_2d_4[[[X, Y], X], X] + \frac{1}{4}d_1d_2d_4[[[X, Y], Y], X] \\
&- \frac{1}{3}d_3d_4[X, [X, Y]] - \frac{1}{3}d_3d_4[Y, [X, Y]] + \frac{1}{2}d_3d_4[[X, Y], X] - \frac{1}{4}d_1d_3d_4[X, [X, [X, Y]]] \\
&- \frac{1}{4}d_1d_3d_4[X, [Y, [X, Y]]] + \frac{1}{2}d_1d_3d_4[X, [[X, Y], X]] - \frac{1}{4}d_1d_3d_4[Y, [X, [X, Y]]] \\
&- \frac{1}{4}d_1d_3d_4[Y, [Y, [X, Y]]] + \frac{1}{2}d_1d_3d_4[Y, [[X, Y], X]] - \frac{1}{4}d_1d_3d_4[[[X, Y], X], X] \\
&+ \frac{1}{4}d_1d_3d_4[[[X, Y], Y], X] - \frac{1}{4}d_2d_3d_4[X, [X, [X, Y]]] - \frac{1}{4}d_2d_3d_4[X, [Y, [X, Y]]] \\
&+ \frac{1}{2}d_2d_3d_4[X, [[X, Y], X]] - \frac{1}{4}d_2d_3d_4[Y, [X, [X, Y]]] - \frac{1}{4}d_2d_3d_4[Y, [Y, [X, Y]]] \\
&+ \frac{1}{2}d_2d_3d_4[Y, [[X, Y], X]] - \frac{1}{4}d_2d_3d_4[[[X, Y], X], X] + \frac{1}{4}d_2d_3d_4[[[X, Y], Y], X] \\
&- \frac{1}{5}d_1d_2d_3d_4[X, [X, [X, [X, Y]]]] - \frac{1}{5}d_1d_2d_3d_4[X, [X, [Y, [X, Y]]]] \\
&+ \frac{1}{2}d_1d_2d_3d_4[X, [X, [[X, Y], X]]] - \frac{1}{5}d_1d_2d_3d_4[X, [Y, [X, [X, Y]]]] \\
&- \frac{1}{5}d_1d_2d_3d_4[X, [Y, [Y, [X, Y]]]] + \frac{1}{2}d_1d_2d_3d_4[X, [Y, [[X, Y], X]]]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{3}d_1d_2d_3d_4[X, [[[X, Y], X], X]] + \frac{1}{3}d_1d_2d_3d_4[X, [[[X, Y], Y], X]] \\
& -\frac{1}{5}d_1d_2d_3d_4[Y, [X, [X, [X, Y]]]] - \frac{1}{5}d_1d_2d_3d_4[Y, [X, [Y, [X, Y]]]] \\
& + \frac{1}{2}d_1d_2d_3d_4[Y, [X, [[X, Y], X]]] - \frac{1}{5}d_1d_2d_3d_4[Y, [Y, [X, [X, Y]]]] \\
& -\frac{1}{5}d_1d_2d_3d_4[Y, [Y, [Y, [X, Y]]]] + \frac{1}{2}d_1d_2d_3d_4[Y, [Y, [[X, Y], X]]] \\
& -\frac{1}{3}d_1d_2d_3d_4[Y, [[[X, Y], X], X]] + \frac{1}{3}d_1d_2d_3d_4[Y, [[[X, Y], Y], X]] \\
& -\frac{1}{2}d_1d_2d_3d_4[[X, Y], [X, [X, Y]]] - \frac{1}{2}d_1d_2d_3d_4[[X, Y], [Y, [X, Y]]] \\
& + d_1d_2d_3d_4[[X, Y], [[X, Y], X]] + \frac{1}{8}d_1d_2d_3d_4[[X, [[X, Y], X]], X] \\
& + \frac{1}{8}d_1d_2d_3d_4[[X, [[X, Y], Y]], X] + \frac{1}{8}d_1d_2d_3d_4[[Y, [[X, Y], X]], X] \\
& + \frac{1}{8}d_1d_2d_3d_4[[Y, [[X, Y], Y]], X] + \frac{1}{3}d_1d_2d_3d_4[[Y, [[X, Y], X]], [X, Y]] \\
& -\frac{1}{3}d_1d_2d_3d_4[[[X, Y], Y], [X, Y]] + \frac{1}{8}d_1d_2d_3d_4[[[[X, Y], X], X], X] \\
& -\frac{1}{8}d_1d_2d_3d_4[[[[X, Y], X], Y], X] - \frac{1}{8}d_1d_2d_3d_4[[[[X, Y], Y], X], X] \\
& + \frac{1}{8}d_1d_2d_3d_4[[[[X, Y], Y], Y], X]
\end{aligned} \tag{26}$$

Letting m_{51} be the right-hand side of (26) with the first term X deleted, we have

$$\begin{aligned}
& (25) \\
& = \exp -d_5 m_{51} \cdot \exp d_5 X + d_5 m_{51} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} \end{aligned} \right) \cdot \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By Proposition 3(} \\
& = \exp -d_5 m_{51} \cdot (X + m_{51})_{d_5} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4)(X + Y) + d_5 Y + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} \end{aligned} \right) \cdot \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By Proposition 4(} \\
& = \exp -d_5 m_{51} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} \end{aligned} \right) \cdot \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By (26)(} \tag{27}
\end{aligned}$$

We let j_{51} be the result of $-m_{51}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, due to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{right}}(\exp) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4[X + Y, [[X, Y], X + Y]] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4[[[X, Y], X - Y], X - Y] \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} \end{aligned} \right) (j_{51}) \\
&= \frac{1}{2}d_1[X, Y] + \frac{1}{2}d_2[X, Y] + \frac{5}{6}d_1 d_2[X, [X, Y]] + \frac{5}{6}d_1 d_2[Y, [X, Y]] - \frac{1}{2}d_1 d_2[[X, Y], X] \\
&+ \frac{1}{2}d_3[X, Y] + \frac{5}{6}d_1 d_3[X, [X, Y]] + \frac{5}{6}d_1 d_3[Y, [X, Y]] - \frac{1}{2}d_1 d_3[[X, Y], X] \\
&+ \frac{5}{6}d_2 d_3[X, [X, Y]] + \frac{5}{6}d_2 d_3[Y, [X, Y]] - \frac{1}{2}d_2 d_3[[X, Y], X] + \frac{5}{4}d_1 d_2 d_3[X, [X, [X, Y]]] \\
&+ \frac{5}{4}d_1 d_2 d_3[X, [Y, [X, Y]]] - \frac{5}{4}d_1 d_2 d_3[X, [[X, Y], X]] + \frac{5}{4}d_1 d_2 d_3[Y, [X, [X, Y]]] \\
&+ \frac{5}{4}d_1 d_2 d_3[Y, [Y, [X, Y]]] - \frac{5}{4}d_1 d_2 d_3[Y, [[X, Y], X]] + \frac{1}{4}d_1 d_2 d_3[[[X, Y], X], X] \\
&- \frac{1}{4}d_1 d_2 d_3[[[X, Y], Y], X] + \frac{1}{2}d_4[X, Y] + \frac{5}{6}d_1 d_4[X, [X, Y]] + \frac{5}{6}d_1 d_4[Y, [X, Y]] \\
&- \frac{1}{2}d_1 d_4[[X, Y], X] + \frac{5}{6}d_2 d_4[X, [X, Y]] + \frac{5}{6}d_2 d_4[Y, [X, Y]] - \frac{1}{2}d_2 d_4[[X, Y], X] \\
&+ \frac{5}{4}d_1 d_2 d_4[X, [X, [X, Y]]] + \frac{5}{4}d_1 d_2 d_4[X, [Y, [X, Y]]] - \frac{5}{4}d_1 d_2 d_4[X, [[X, Y], X]] \\
&+ \frac{5}{4}d_1 d_2 d_4[Y, [X, [X, Y]]] + \frac{5}{4}d_1 d_2 d_4[Y, [Y, [X, Y]]] - \frac{5}{4}d_1 d_2 d_4[Y, [[X, Y], X]] \\
&+ \frac{1}{4}d_1 d_2 d_4[[[X, Y], X], X] - \frac{1}{4}d_1 d_2 d_4[[[X, Y], Y], X] + \frac{5}{6}d_3 d_4[X, [X, Y]] \\
&+ \frac{5}{6}d_3 d_4[Y, [X, Y]] - \frac{1}{2}d_3 d_4[[X, Y], X] + \frac{5}{4}d_1 d_3 d_4[X, [X, [X, Y]]] \\
&+ \frac{5}{4}d_1 d_3 d_4[X, [Y, [X, Y]]] - \frac{5}{4}d_1 d_3 d_4[X, [[X, Y], X]] + \frac{5}{4}d_1 d_3 d_4[Y, [X, [X, Y]]] \\
&+ \frac{5}{4}d_1 d_3 d_4[Y, [Y, [X, Y]]] - \frac{5}{4}d_1 d_3 d_4[Y, [[X, Y], X]] + \frac{1}{4}d_1 d_3 d_4[[[X, Y], X], X] \\
&- \frac{1}{4}d_1 d_3 d_4[[[X, Y], Y], X] + \frac{5}{4}d_1 d_3 d_4[X, [X, [X, Y]]] + \frac{5}{4}d_1 d_3 d_4[X, [Y, [X, Y]]] \\
&- \frac{5}{4}d_1 d_3 d_4[X, [[X, Y], X]] + \frac{5}{4}d_2 d_3 d_4[Y, [X, [X, Y]]] + \frac{5}{4}d_2 d_3 d_4[Y, [Y, [X, Y]]] \\
&- \frac{5}{4}d_2 d_3 d_4[Y, [[X, Y], X]] + \frac{1}{4}d_2 d_3 d_4[[[X, Y], X], X] - \frac{1}{4}d_2 d_3 d_4[[[X, Y], Y], X] \\
&+ \frac{5}{3}d_1 d_2 d_3 d_4[X, [X, [X, [X, Y]]]] + \frac{5}{3}d_1 d_2 d_3 d_4[X, [X, [Y, [X, Y]]]] \\
&- 2d_1 d_2 d_3 d_4[X, [X, [[X, Y], X]]] + \frac{5}{3}d_1 d_2 d_3 d_4[X, [Y, [X, [X, Y]]]] \\
&+ \frac{5}{3}d_1 d_2 d_3 d_4[X, [Y, [Y, [X, Y]]]] - 2d_1 d_2 d_3 d_4[X, [Y, [[X, Y], X]]]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}d_1d_2d_3d_4[X, [[[X, Y], X], X]] - \frac{1}{2}d_1d_2d_3d_4[X, [[[X, Y], Y], X]] \\
& + \frac{5}{3}d_1d_2d_3d_4[Y, [X, [X, [X, Y]]]] + \frac{5}{3}d_1d_2d_3d_4[Y, [X, [Y, [X, Y]]]] \\
& - 2d_1d_2d_3d_4[Y, [X, [[X, Y], X]]] + \frac{5}{3}d_1d_2d_3d_4[Y, [Y, [X, [X, Y]]]] \\
& + \frac{5}{3}d_1d_2d_3d_4[Y, [Y, [Y, [X, Y]]]] - 2d_1d_2d_3d_4[Y, [Y, [[X, Y], X]]] \\
& + \frac{1}{2}d_1d_2d_3d_4[Y, [[[X, Y], X], X]] - \frac{1}{2}d_1d_2d_3d_4[Y, [[[X, Y], Y], X]] \\
& + 2d_1d_2d_3d_4[[X, Y], [X, [X, Y]]] + 2d_1d_2d_3d_4[[X, Y], [Y, [X, Y]]] \\
& - \frac{3}{2}d_1d_2d_3d_4[[X, Y], [[X, Y], X]] - \frac{1}{2}d_1d_2d_3d_4[[[X, Y], X], [X, Y]] \\
& + \frac{1}{2}d_1d_2d_3d_4[[[X, Y], Y], [X, Y]]
\end{aligned} \tag{28}$$

Letting m_{52} be the right-hand side of (28), we have

$$\begin{aligned}
& (27) \\
& = \exp -d_5m_{51} - d_5m_{52} \cdot \exp d_5m_{52} \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5i_{51} + d_5i_{52} + d_5i_{53} \end{aligned} \right) \cdot \\
& \exp -d_5n_{51} - d_5n_{52} - d_5n_{53} - d_5n_{54} \\
&) \text{By Proposition 3(} \\
& = \exp -d_5m_{51} - d_5m_{52} \cdot (m_{52})_{d_5} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5i_{51} + d_5i_{52} + d_5i_{53} \end{aligned} \right) \cdot \\
& \exp -d_5n_{51} - d_5n_{52} - d_5n_{53} - d_5n_{54} \\
&) \text{By Proposition 4(} \\
& = \exp -d_5m_{51} - d_5m_{52} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5i_{51} + d_5i_{52} + d_5i_{53} + d_5j_{51} \end{aligned} \right) \cdot \\
& \exp -d_5n_{51} - d_5n_{52} - d_5n_{53} - d_5n_{54} \\
&) \text{By (28)(} \tag{29}
\end{aligned}$$

We let j_{52} be the result of $-m_{51} - m_{52}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, by dint of Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{right}}(\exp) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2[X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3[[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4[X + Y, [[X, Y], X + Y]] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4[[[X, Y], X - Y], X - Y] \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} \end{aligned} \right) (j_{52}) \\
&= -\frac{1}{2}d_1 d_2[X, [X, Y]] - \frac{1}{2}d_1 d_2[Y, [X, Y]] - \frac{1}{2}d_1 d_3[X, [X, Y]] - \frac{1}{2}d_1 d_3[Y, [X, Y]] \\
&- \frac{1}{2}d_2 d_3[X, [X, Y]] - \frac{1}{2}d_2 d_3[Y, [X, Y]] - \frac{7}{4}d_1 d_2 d_3[X, [X, [X, Y]]] \\
&- \frac{7}{4}d_1 d_2 d_3[X, [Y, [X, Y]]] + \frac{3}{4}d_1 d_2 d_3[X, [[X, Y], X]] - \frac{7}{4}d_1 d_2 d_3[Y, [X, [X, Y]]] \\
&- \frac{7}{4}d_1 d_2 d_3[Y, [Y, [X, Y]]] + \frac{3}{4}d_1 d_2 d_3[Y, [[X, Y], X]] - \frac{1}{2}d_1 d_4[X, [X, Y]] \\
&- \frac{1}{2}d_1 d_4[Y, [X, Y]] - \frac{1}{2}d_2 d_4[X, [X, Y]] - \frac{1}{2}d_2 d_4[Y, [X, Y]] - \frac{7}{4}d_1 d_2 d_4[X, [X, [X, Y]]] \\
&- \frac{7}{4}d_1 d_2 d_4[X, [Y, [X, Y]]] + \frac{3}{4}d_1 d_2 d_4[X, [[X, Y], X]] - \frac{7}{4}d_1 d_2 d_4[Y, [X, [X, Y]]] \\
&- \frac{7}{4}d_1 d_2 d_4[Y, [Y, [X, Y]]] + \frac{3}{4}d_1 d_2 d_4[Y, [[X, Y], X]] - \frac{1}{2}d_3 d_4[X, [X, Y]] \\
&- \frac{1}{2}d_3 d_4[Y, [X, Y]] - \frac{7}{4}d_1 d_3 d_4[X, [X, [X, Y]]] - \frac{7}{4}d_1 d_3 d_4[X, [Y, [X, Y]]] \\
&+ \frac{3}{4}d_1 d_3 d_4[X, [[X, Y], X]] - \frac{7}{4}d_1 d_3 d_4[Y, [X, [X, Y]]] - \frac{7}{4}d_1 d_3 d_4[Y, [Y, [X, Y]]] \\
&+ \frac{3}{4}d_1 d_3 d_4[Y, [[X, Y], X]] - \frac{7}{4}d_2 d_3 d_4[X, [X, [X, Y]]] - \frac{7}{4}d_2 d_3 d_4[X, [Y, [X, Y]]] \\
&+ \frac{3}{4}d_2 d_3 d_4[X, [[X, Y], X]] - \frac{7}{4}d_2 d_3 d_4[Y, [X, [X, Y]]] - \frac{7}{4}d_2 d_3 d_4[Y, [Y, [X, Y]]] \\
&+ \frac{3}{4}d_2 d_3 d_4[Y, [[X, Y], X]] - 3d_1 d_2 d_3 d_4[X, [X, [X, [X, Y]]]] \\
&- 3d_1 d_2 d_3 d_4[X, [X, [Y, [X, Y]]]] + \frac{1}{2}d_1 d_2 d_3 d_4[X, [X, [[X, Y], X]]] \\
&- 3d_1 d_2 d_3 d_4[X, [Y, [X, [X, Y]]]] - 3d_1 d_2 d_3 d_4[X, [Y, [Y, [X, Y]]]] \\
&+ \frac{1}{2}d_1 d_2 d_3 d_4[X, [Y, [[X, Y], X]]] - 3d_1 d_2 d_3 d_4[Y, [X, [X, [X, Y]]]] \\
&- 3d_1 d_2 d_3 d_4[Y, [X, [Y, [X, Y]]]] - 3d_1 d_2 d_3 d_4[Y, [Y, [X, [X, Y]]]] \\
&- 3d_1 d_2 d_3 d_4[Y, [Y, [Y, [X, Y]]]] + \frac{1}{2}d_1 d_2 d_3 d_4[Y, [X, [[X, Y], X]]] \\
&+ \frac{1}{2}d_1 d_2 d_3 d_4[Y, [Y, [[X, Y], X]]] - \frac{3}{2}d_1 d_2 d_3 d_4[[X, Y], [X, [X, Y]]] \\
&- \frac{3}{2}d_1 d_2 d_3 d_4[[X, Y], [Y, [X, Y]]] \tag{30}
\end{aligned}$$

Letting m_{53} be the right-hand side of (30), we have

$$\begin{aligned}
& (29) \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} \cdot \exp d_5 m_{53} \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} \end{aligned} \right) \cdot \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By Proposition 3(} \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} \cdot (m_{53})_{d_5} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} \end{aligned} \right) \cdot \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By Proposition 4(} \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} + d_5 j_{52} \end{aligned} \right) \cdot \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By (30)(} \tag{31}
\end{aligned}$$

We let j_{53} be the result of $-m_{51} - m_{52} - m_{53}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, thanks to Theorem 2, we have

$$\begin{aligned}
& \delta^{\text{right}}(\exp) \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 [X + Y, [[X, Y], X + Y]] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 [[[X, Y], X - Y], X - Y] \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} + d_5 j_{52} \end{aligned} \right) (j_{53}) \\
&= \frac{3}{4} d_1 d_2 d_3 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [Y, [X, [X, Y]]] \\
&+ \frac{3}{4} d_1 d_2 d_3 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [Y, [X, Y]]] \\
&+ \frac{3}{4} d_1 d_2 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [X, [X, [X, Y]]] \\
&+ \frac{3}{4} d_1 d_3 d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [Y, [Y, [X, Y]]] \\
&+ \frac{3}{4} d_2 d_3 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]] \\
&+ \frac{3}{4} d_2 d_3 d_4 [Y, [Y, [X, Y]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [X, [X, Y]]]] \\
&+ \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [X, [Y, [X, [X, Y]]]] \\
&+ \frac{1}{2} d_1 d_2 d_3 d_4 [X, [Y, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [X, [X, Y]]]] \\
&+ \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [Y, [X, [X, Y]]]] \\
&+ \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [Y, [Y, [X, Y]]]] \tag{32}
\end{aligned}$$

Letting m_{54} be the right-hand side of (32), we have

$$\begin{aligned}
& (31) \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54} \cdot \exp d_5 m_{54} \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} + d_5 j_{52} \end{aligned} \right) \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By Proposition 3(} \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54} \cdot (m_{54})_{d_5} \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} + d_5 j_{52} \end{aligned} \right) \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By Proposition 4(} \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54} \cdot \\
& \exp \left(\begin{aligned} & (d_1 + d_2 + d_3 + d_4 + d_5)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ & + \frac{1}{12}(d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ & + \frac{1}{96}(d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]) \\ & + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} + d_5 j_{52} + d_5 j_{53} \end{aligned} \right) \\
& \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
&) \text{By (32)(} \tag{33}
\end{aligned}$$

Since the coefficient of every term in $-m_{51} - m_{52} - m_{53} - m_{54}$ contains $d_1 d_2 d_3 d_4$, we are done. We have

$$\begin{aligned}
& i_{51} + i_{52} + i_{53} + j_{51} + j_{52} + j_{53} \\
&= d_1 [X, Y] + d_2 [X, Y] - \frac{1}{2} d_1 d_2 [[X, Y], X] + \frac{1}{2} d_1 d_2 [[X, Y], Y] + d_3 [X, Y] \\
&- \frac{1}{2} d_1 d_3 [[X, Y], X] + \frac{1}{2} d_1 d_3 [[X, Y], Y] - \frac{1}{2} d_2 d_3 [[X, Y], X] \\
&+ \frac{1}{2} d_2 d_3 [[X, Y], Y] + \frac{1}{4} d_1 d_2 d_3 [X, [[X, Y], X]] + \frac{1}{4} d_1 d_2 d_3 [X, [[X, Y], Y]] \\
&+ \frac{1}{4} d_1 d_2 d_3 [Y, [[X, Y], X]] + \frac{1}{4} d_1 d_2 d_3 [Y, [[X, Y], Y]] + \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], X] \\
&- \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], Y] - \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], X] + \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], Y] \\
&+ d_4 [X, Y] - \frac{1}{2} d_1 d_4 [[X, Y], X] + \frac{1}{2} d_1 d_4 [[X, Y], Y] - \frac{1}{2} d_2 d_4 [[X, Y], X] \\
&+ \frac{1}{2} d_2 d_4 [[X, Y], Y] + \frac{1}{4} d_1 d_2 d_4 [X, [[X, Y], X]] + \frac{1}{4} d_1 d_2 d_4 [X, [[X, Y], Y]] \\
&+ \frac{1}{4} d_1 d_2 d_4 [Y, [[X, Y], X]] + \frac{1}{4} d_1 d_2 d_4 [Y, [[X, Y], Y]] + \frac{1}{4} d_1 d_2 d_4 [[[X, Y], X], X] \\
&- \frac{1}{4} d_1 d_2 d_4 [[[X, Y], X], Y] - \frac{1}{4} d_1 d_2 d_4 [[[X, Y], Y], X] + \frac{1}{4} d_1 d_2 d_4 [[[X, Y], Y], Y] \\
&- \frac{1}{2} d_3 d_4 [[X, Y], X] + \frac{1}{2} d_3 d_4 [[X, Y], Y] + \frac{1}{4} d_1 d_3 d_4 [X, [[X, Y], X]] \\
&+ \frac{1}{4} d_1 d_3 d_4 [X, [[X, Y], Y]] + \frac{1}{4} d_1 d_3 d_4 [Y, [[X, Y], X]] + \frac{1}{4} d_1 d_3 d_4 [Y, [[X, Y], Y]] \\
&+ \frac{1}{4} d_1 d_3 d_4 [[[X, Y], X], X] - \frac{1}{4} d_1 d_3 d_4 [[[X, Y], X], Y] - \frac{1}{4} d_1 d_3 d_4 [[[X, Y], Y], X] \\
&+ \frac{1}{4} d_1 d_3 d_4 [[[X, Y], Y], Y] + \frac{1}{4} d_2 d_3 d_4 [X, [[X, Y], X]] + \frac{1}{4} d_2 d_3 d_4 [X, [[X, Y], Y]] \\
&+ \frac{1}{4} d_2 d_3 d_4 [Y, [[X, Y], X]] + \frac{1}{4} d_2 d_3 d_4 [Y, [[X, Y], Y]] + \frac{1}{4} d_2 d_3 d_4 [[[X, Y], X], X] \\
&- \frac{1}{4} d_2 d_3 d_4 [[[X, Y], X], Y] - \frac{1}{4} d_2 d_3 d_4 [[[X, Y], Y], X] + \frac{1}{4} d_2 d_3 d_4 [[[X, Y], Y], Y]
\end{aligned}$$

on the one hand, and

$$\begin{aligned}
& -m_{51} - m_{52} - m_{53} - m_{54} - n_{51} - n_{52} - n_{53} - n_{54} \\
& = d_1 d_2 d_3 d_4 [X, [X, [[X, Y], X]]] - d_1 d_2 d_3 d_4 [X, [X, [[X, Y], Y]]] \\
& + d_1 d_2 d_3 d_4 [X, [Y, [[X, Y], X]]] - d_1 d_2 d_3 d_4 [X, [Y, [[X, Y], Y]]] \\
& + d_1 d_2 d_3 d_4 [Y, [X, [[X, Y], X]]] - d_1 d_2 d_3 d_4 [Y, [X, [[X, Y], Y]]] \\
& + d_1 d_2 d_3 d_4 [Y, [Y, [[X, Y], X]]] - d_1 d_2 d_3 d_4 [Y, [Y, [[X, Y], Y]]] \\
& - \frac{1}{6} d_1 d_2 d_3 d_4 [X, [[[X, Y], X], X]] - \frac{1}{6} d_1 d_2 d_3 d_4 [X, [[[X, Y], X], Y]] \\
& + \frac{1}{6} d_1 d_2 d_3 d_4 [X, [[[X, Y], Y], X]] + \frac{1}{6} d_1 d_2 d_3 d_4 [X, [[[X, Y], Y], Y]] \\
& - \frac{1}{6} d_1 d_2 d_3 d_4 [Y, [[[X, Y], X], X]] - \frac{1}{6} d_1 d_2 d_3 d_4 [Y, [[[X, Y], X], Y]] \\
& + \frac{1}{6} d_1 d_2 d_3 d_4 [Y, [[[X, Y], Y], X]] + \frac{1}{6} d_1 d_2 d_3 d_4 [Y, [[[X, Y], Y], Y]] \\
& + \frac{1}{2} d_1 d_2 d_3 d_4 [[X, Y], [[X, Y], X]] + \frac{1}{2} d_1 d_2 d_3 d_4 [[X, Y], [[X, Y], Y]] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 [[X, [[X, Y], X]], X] + \frac{1}{8} d_1 d_2 d_3 d_4 [[X, [[X, Y], X]], Y] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 [[X, [[X, Y], Y]], X] + \frac{1}{8} d_1 d_2 d_3 d_4 [[X, [[X, Y], Y]], Y] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 [[Y, [[X, Y], X]], X] + \frac{1}{8} d_1 d_2 d_3 d_4 [[Y, [[X, Y], X]], Y] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 [[Y, [[X, Y], Y]], X] + \frac{1}{8} d_1 d_2 d_3 d_4 [[Y, [[X, Y], Y]], Y] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 [[[[X, Y], X], X], X] + \frac{1}{8} d_1 d_2 d_3 d_4 [[[[X, Y], X], X], Y] \\
& + \frac{1}{8} d_1 d_2 d_3 d_4 [[[[X, Y], X], Y], X] - \frac{1}{8} d_1 d_2 d_3 d_4 [[[[X, Y], X], Y], Y] \\
& + \frac{1}{8} d_1 d_2 d_3 d_4 [[[[X, Y], Y], X], X] - \frac{1}{8} d_1 d_2 d_3 d_4 [[[[X, Y], Y], X], Y] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 [[[[X, Y], Y], Y], X] + \frac{1}{8} d_1 d_2 d_3 d_4 [[[[X, Y], Y], Y], Y]
\end{aligned}$$

on the other. Therefore we have the desired result. ■

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